General Sol'n to transients in an LRC series circuit

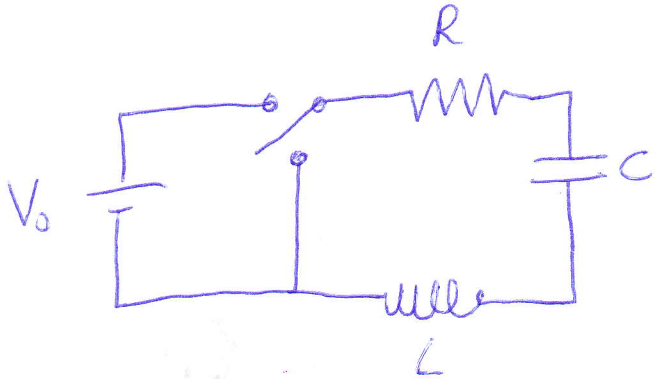

Fig. 1

K.V.R.

$$V_0 - IR - \frac{q}{C} - L\frac{dI}{dt} = 0$$

$$\therefore V_0 = L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{1}{C}q$$

① First, assume a sol'n of the form

(A) $q(t) = q_h(t) + q_p$ where q_p is indep. of time.

then

$$\underline{V_0} = L\frac{d^2q_h(t)}{dt^2} + R\frac{dq_h(t)}{dt} + \frac{1}{C}q_h(t) + \underline{\frac{1}{C}q_p}$$

Time indep. parts must be equal:

$$\therefore \boxed{q_p = CV_0}$$

2) Remaining time-dependent part is:

$$0 = L\frac{d^2q_n(t)}{dt^2} + R\frac{dq_n(t)}{dt} + \frac{1}{C}q_n(t) \quad \text{homogeneous diff. eq'n.}$$

$$= \frac{d^2q_n(t)}{dt^2} + \frac{R}{L}\frac{dq_n(t)}{dt} + \frac{1}{LC}q_n(t)$$

define $\gamma = \frac{R}{L}$ $\quad \omega_0^2 = \frac{1}{LC}$

$$0 = \frac{d^2q_n(t)}{dt^2} + \gamma\frac{dq_n(t)}{dt} + \omega_0^2 q_n(t)$$

(i) $$\therefore \underbrace{\left(\frac{d^2}{dt^2} + \gamma\frac{d}{dt} + \omega_0^2\right)}_{\text{like quadratic formula.}} q_n(t) = 0$$

Aside. if $\underbrace{ax^2+bx+c=0}_{\text{can be rewritten as:}}$ then $x = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$

$$= \frac{-b}{2a} \pm \sqrt{\left(\frac{b}{2a}\right)^2 - \frac{c}{a}}$$

$$\left(x + \frac{b}{2a} - \sqrt{\left(\frac{b}{2a}\right)^2 - \frac{c}{a}}\right)\left(x + \frac{b}{2a} + \sqrt{\left(\frac{b}{2a}\right)^2 - \frac{c}{a}}\right) = 0$$

check

$$x^2 + \frac{b}{a}x + \cancel{\left(\frac{b}{2a}\right)^2} - \left[\cancel{\left(\frac{b}{2a}\right)^2} - \frac{c}{a}\right] = 0$$

$$\therefore ax^2 + bx + c = 0 \quad \checkmark$$

Apply same factoring technique to the $\left(\frac{d^2}{dt^2} + \gamma\frac{d}{dt} + \omega_0^2\right)$ operator.

$a \rightarrow 1$ $\quad x \rightarrow \frac{d}{dt}$

$b \rightarrow \gamma$

$c \rightarrow \omega_0^2$

(i) becomes

(ii) $$\Bigg(\frac{d}{dt} + \frac{\gamma}{2} - \underbrace{\sqrt{\left(\frac{\gamma}{2}\right)^2 - \omega_0^2}}_{\equiv \beta}\Bigg)\Bigg(\frac{d}{dt} + \frac{\gamma}{2} + \underbrace{\sqrt{\left(\frac{\gamma}{2}\right)^2 - \omega_0^2}}_{\equiv \beta}\Bigg) q_h(t) = 0$$

3 Now assume that can find $q_1(t)$ & $q_2(t)$ such that:

(iii) $$\left(\frac{d}{dt} + \frac{\gamma}{2} - \beta\right) q_1(t) = 0$$

(iv) $$\left(\frac{d}{dt} + \frac{\gamma}{2} + \beta\right) q_2(t) = 0$$

Let's try $q_h(t) = q_1(t) + q_2(t)$ in Eq'n (ii)

$$\left(\frac{d}{dt} + \frac{\gamma}{2} - \beta\right)\left(\frac{d}{dt} + \frac{\gamma}{2} + \beta\right)\left(q_1(t) + q_2(t)\right)$$

$$= \left(\frac{d}{dt} + \frac{\gamma}{2} + \beta\right)\underbrace{\left(\frac{d}{dt} + \frac{\gamma}{2} - \beta\right) q_1(t)}_{=0 \text{ by (iii)}} + \left(\frac{d}{dt} + \frac{\gamma}{2} - \beta\right)\underbrace{\left(\frac{d}{dt} + \frac{\gamma}{2} + \beta\right) q_2(t)}_{0 \text{ by (iv)}} = 0$$

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$\therefore$ If (iii) & (iv) can be solved, then know that

$q_h(t) = q_1(t) + q_2(t)$ is a sol'n to (i)

(4) Start w/ (iii) $\left(\frac{d}{dt} + \frac{\gamma}{2} - \beta\right) q_1(t) = 0$

$$\therefore \frac{d}{dt} q_1(t) = -\left(\frac{\gamma}{2} - \beta\right) q_1(t)$$

$$\therefore \frac{dq_1(t)}{q_1(t)} = -\left(\frac{\gamma}{2} - \beta\right) dt$$

integrate

integration const.

$$\ln q_1(t) = -\left(\frac{\gamma}{2} - \beta\right) t + B$$

$$\therefore q_1(t) = \underbrace{e^{B}}_{\equiv A_1} e^{-\left(\frac{\gamma}{2} - \beta\right) t}$$

(v) $$\boxed{q_1(t) = A_1 e^{-\left(\frac{\gamma}{2} - \beta\right) t}}$$

Exact same procedure applied to (iv) gives:

(vi) $$\boxed{q_2(t) = A_2 e^{-\left(\frac{\gamma}{2} + \beta\right) t}}$$

5 ∴ We get $q_h(t) = q_1(t) + q_2(t)$

$$= \left(A_1 e^{\beta t} + A_2 e^{-\beta t}\right) e^{-\frac{\gamma}{2} t}$$

We need to use initial conditions to find A_1 & A_2.

For this example, we'll assume that $\beta^2 = \left(\frac{\gamma}{2}\right)^2 - \omega_0^2 < 0$

Recall $\gamma = \frac{R}{L}$ $\omega_0^2 = \frac{1}{LC}$

∴ assuming that $\left(\frac{R}{2L}\right)^2 - \frac{1}{LC} < 0 \Rightarrow$ R relatively small.

$\beta^2 < 0$ case referred to as the $\boxed{\text{underdamped}}$ case

Now $\beta = \sqrt{\left(\frac{\gamma}{2}\right)^2 - \omega_0^2} = \underbrace{\sqrt{-1}}_{j} \underbrace{\sqrt{\omega_0^2 - \left(\frac{\gamma}{2}\right)^2}}_{\equiv \omega_1 \text{ (positive)}} = j\omega_1$

$$\therefore q_h(t) = \left(A_1 e^{j\omega_1 t} + A_2 e^{-j\omega_1 t}\right) e^{-\gamma/2\, t}$$

Recall Euler eq'n $e^{\pm j\phi} = \cos\phi \pm j\sin\phi$

$$\therefore q_n(t) = \left[A_1(\cos\omega_1 t + j\sin\omega_1 t) + A_2(\cos\omega_1 t - j\sin\omega_1 t)\right]e^{-\gamma/2\, t}$$

$$= \left[(A_1+A_2)\cos\omega_1 t + j(A_1-A_2)\sin\omega_1 t\right]e^{-\frac{\gamma}{2}t}$$

Clearly $q_n(t)$ is a real quantity. $\therefore$ must have

$A_1 - A_2 = 0$ or $A_1 = A_2 \equiv A$

$$\therefore \boxed{q_n(t) = 2Ae^{-\frac{\gamma}{2}t}\cos\omega_1 t}$$

still need to determine A!

(6) Returning to Fig. 1, let's assume that initially switch was in down position for a long time so that initially capacitor is uncharged.

Then at $t=0$ flip switch to up position.

@ $t=0$ $V_c = \frac{q}{C} = 0$ i.e. @ $t=0$ $q=0$

Remember that the full sol'n for q is given by ④

$$q(t) = q_h(t) + q_p$$
$$= 2Ae^{-\frac{\gamma}{2}t}\cos\omega_1 t + CV_0$$

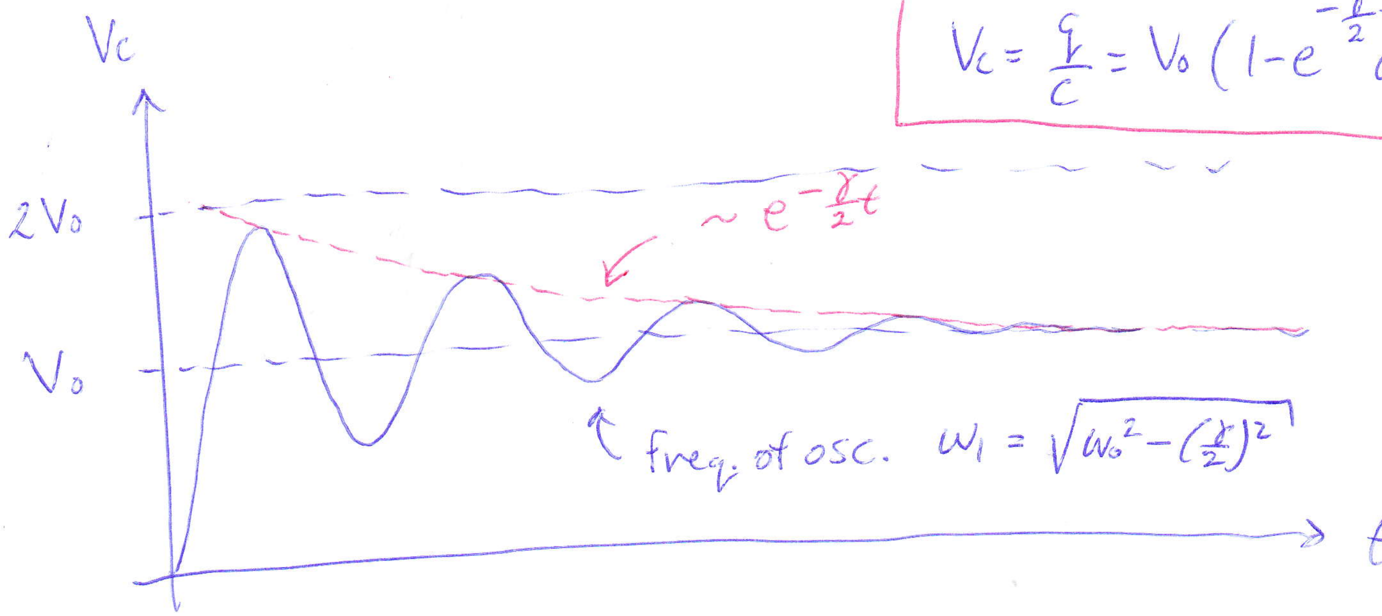
$\therefore$ @ $t=0$ must have

$$2A + CV_0 = 0 \quad \therefore A = -\frac{CV_0}{2}$$

So that finally

(vii) $$\boxed{q(t) = CV_0\left(1 - e^{-\frac{\gamma}{2}t}\cos\omega_1 t\right)}$$

$$\boxed{V_c = \frac{q}{C} = V_0\left(1 - e^{-\frac{\gamma}{2}t}\cos\omega_1 t\right)}$$



Note that Voltage across capacitor, at some instances in time, exceeds the voltage of the battery!

Let's try to understand this by considering the voltage across the inductor.

$$V_L = L\frac{dI}{dt} = L\frac{d^2q}{dt^2}$$

from (vii) $$\boxed{I = \frac{dq}{dt} = CV_0\frac{\gamma}{2}e^{-\frac{\gamma}{2}t}\cos\omega_1 t + CV_0 e^{-\frac{\gamma}{2}t}\omega_1\sin\omega_1 t}$$

$$\frac{d^2q}{dt^2} = -CV_0\left(\frac{\gamma}{2}\right)^2 e^{-\frac{\gamma}{2}t}\cos\omega_1 t - CV_0\frac{\gamma}{2}e^{-\frac{\gamma}{2}t}\omega_1\sin\omega_1 t$$
$$- CV_0\frac{\gamma}{2}e^{-\frac{\gamma}{2}t}\omega_1\sin\omega_1 t + CV_0 e^{-\frac{\gamma}{2}t}\omega_1^2\cos\omega_1 t$$

$$= -CV_0\gamma\omega_1 e^{-\frac{\gamma}{2}t}\sin\omega_1 t$$
$$+ CV_0 e^{-\frac{\gamma}{2}t}\cos\omega_1 t\left[\omega_1^2 - \left(\frac{\gamma}{2}\right)^2\right]$$

$$\therefore \boxed{V_L = L\frac{d^2q}{dt^2} = \frac{-V_0\gamma\omega_1}{\omega_0^2}e^{-\frac{\gamma}{2}t}\sin\omega_1 t + \frac{V_0\left[\omega_1^2 - \left(\frac{\gamma}{2}\right)^2\right]}{\omega_0^2}e^{-\frac{\gamma}{2}t}\cos\omega_1 t}$$

$$\boxed{V_R = IR}$$

☆ Complicated! ☆

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For example, plot V_C, V_L, & V_R as a fcn of time using values from Lab 4.

$V_0 = 5\,V$

$R = 1000\,\Omega$

$L = 20\,mH$

$C = 1\,nF$

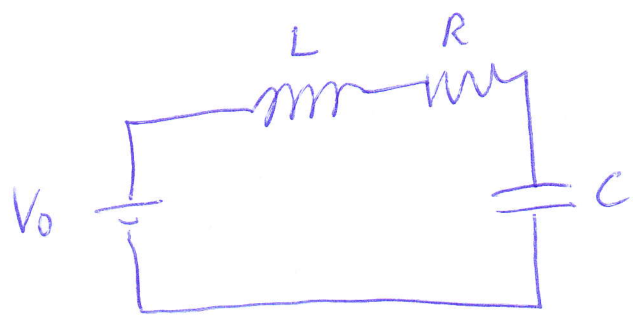
$$\gamma = \frac{R}{L} = 5\times10^{4}\,s^{-1}$$

$$\omega_0^2 = \frac{1}{LC} = 5\times10^{10}\,s^{-2} \Rightarrow \omega_0 = 2.24\times10^{5}\,s^{-1}$$

$$\therefore\ \omega_1 = \sqrt{\omega_0^2 - \left(\frac{\gamma}{2}\right)^2} = 2.22\times10^{5}\,s^{-1} \approx \omega_0 .$$

See plots on the next page.

When switch is closed, have following circuit



From the plots can clearly see that

$V_0 = V_R + V_L + V_C$ s.t. Kirchhoff loop rule satisfied at all times.

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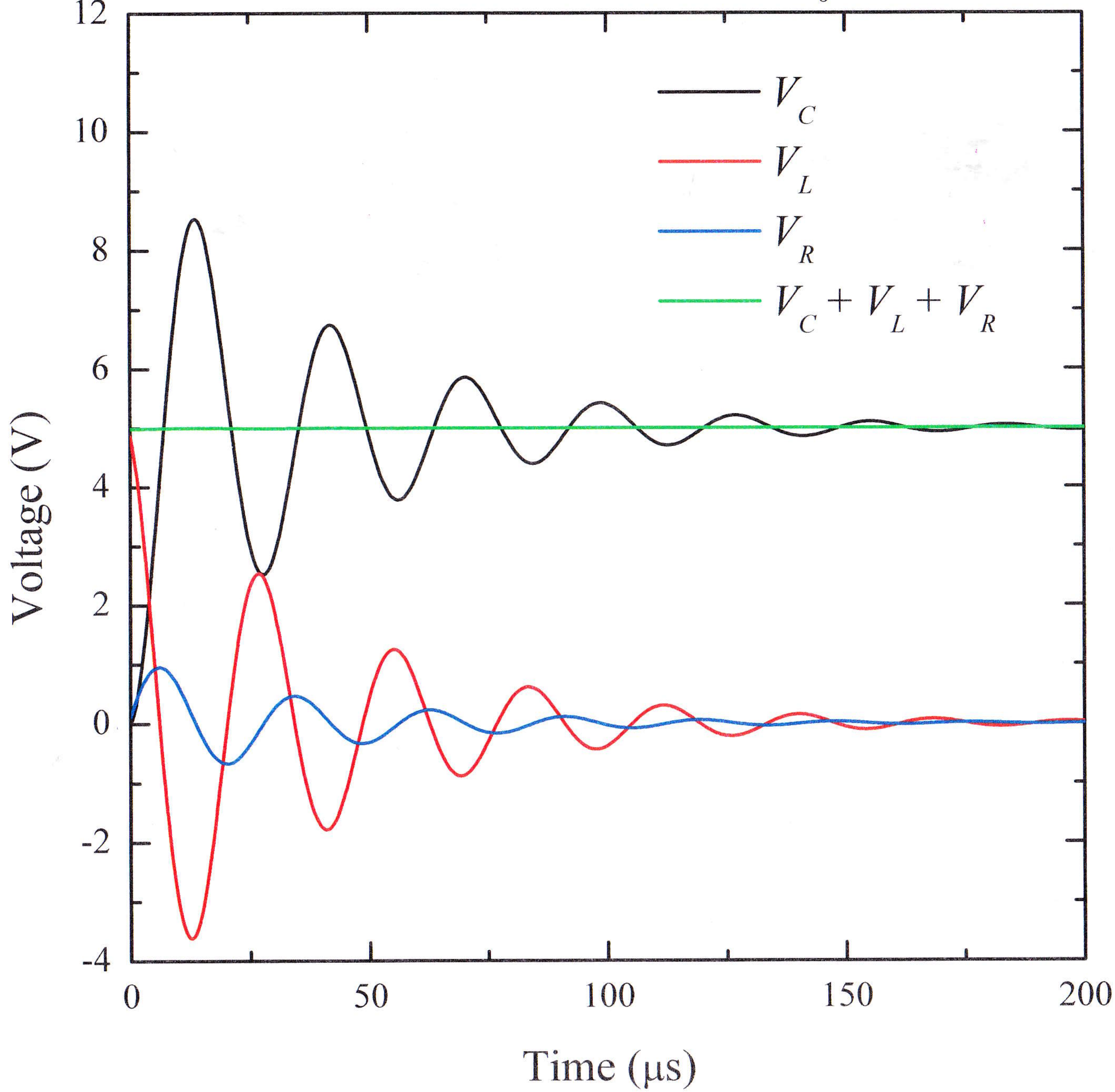
Series *LRC*-circuit for an initially uncharged capacitor with:
$R = 1\ \mathrm{k\Omega}$, $L = 20\ \mathrm{mH}$, $C = 1\ \mathrm{nF}$, and $V_0 = 5\ \mathrm{V}$



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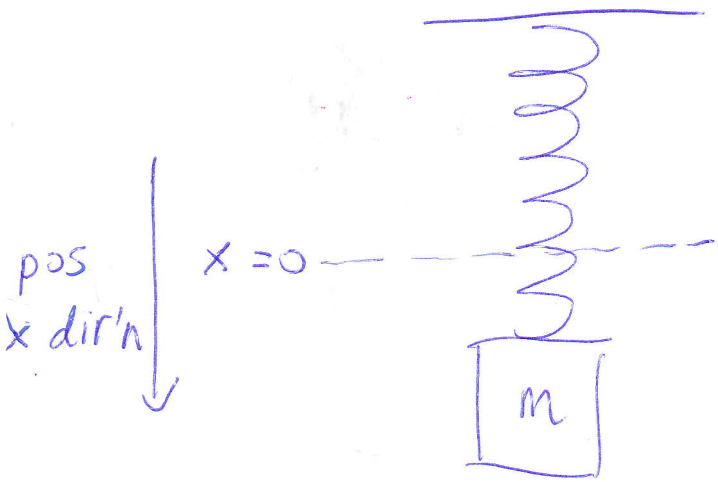
Consider the mechanical analogy of mass on a spring experiencing viscous drag



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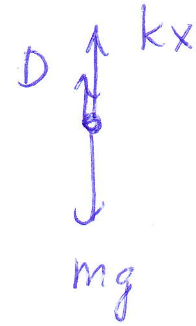

assume

$D = bv$

(prop. to v)

$$ma = -kx - bv + mg$$

$$\therefore \quad m\frac{d^2x}{dt^2} + b\frac{dx}{dt} + kx = mg$$

$$\text{or} \quad \frac{d^2x}{dt^2} + \frac{b}{m}\frac{dx}{dt} + \frac{k}{m}x = g$$

similar to our original diff. eq'n for LRC series circuit

$$L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{1}{C}q = V_0$$

$$\text{or} \quad \frac{d^2q}{dt^2} + \frac{R}{L}\frac{dq}{dt} + \frac{1}{LC}q = \frac{V_0}{L}$$

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If you had a mass suspended from a spring & you could suddenly change the value of the gravitational acceleration g,

then mass would start oscillating about a new equilibrium position $\left(x_{eq} = \frac{mg}{k} \right)$.

Of course, the instantaneous position could be greater than or less than x_{eq}.

For the LRC circuit, the equilibrium charge on the capacitor is $q_{eq} = C V_0$ $\left(= q_P \text{ on page 1} \right)$

If we suddenly make a change to the value of V_0, charge will oscillate, but around a new equilibrium position.

- Let's push the analogy a little further...

For the mass on a spring, if spring is stretched beyond x_{eq} in pos. dir'n then acceleration $\frac{d^2x}{dt^2}$ is in neg. dir'n. (and vice-versa).

Likewise, for the LRC circuit, if capacitor charge is greater than q_{eq}, then $\frac{d^2q}{dt^2} \propto V_L$ is negative.

Take a look at the plots on page 10.

- Also, for mass on a spring, $v = \frac{dx}{dt}$ is max as mass passes through x_{eq} & $\frac{dx}{dt} = 0$ when mass is a turning point

Likewise, for LRC circuit, if capacitor charge is passing through q_{eq}, then $\frac{dq}{dt} \propto V_R$ is a maximum.

$\frac{dq}{dt} \propto V_R$ is zero when q is a max or min.

Look at the plots on page 10.

Note $q \propto V_C$

As a last point, consider the energy.

Energy stored in capacitor is $\boxed{U_c = \frac{1}{2C} q^2}$

Energy stored in inductor is $\boxed{U_L = \frac{L}{2} I^2}$

Know $q(t) = CV_0\left(1 - e^{-\frac{\gamma}{2}t} \cos \omega_1 t\right)$

$$I(t) = CV_0 e^{-\frac{\gamma}{2}t}\left[\frac{\gamma}{2}\cos \omega_1 t + \omega_1 \sin \omega_1 t\right]$$

Using the same component values on page 9,

can plot U_c & U_L as a fcn of time.

The <u>power</u> (energy per unit time) dissipated by the resistor is $I^2 R$

$$\boxed{U_R = \int_0^t I^2 R \, dt}$$

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The sum of the energy stored in C, L, and energy consumed by R, should total the energy provided by the battery (voltage source).

Power provided by battery (source) is

$$P_{source} = I \cdot V$$

$$\therefore \boxed{U_{source} = \int_0^t I V \, dt}$$

The plot on page 16 shows U_R, U_L, U_C & verifies that $U_R + U_L + U_C = U_{source}$

(energy is conserved at <u>all</u> times)

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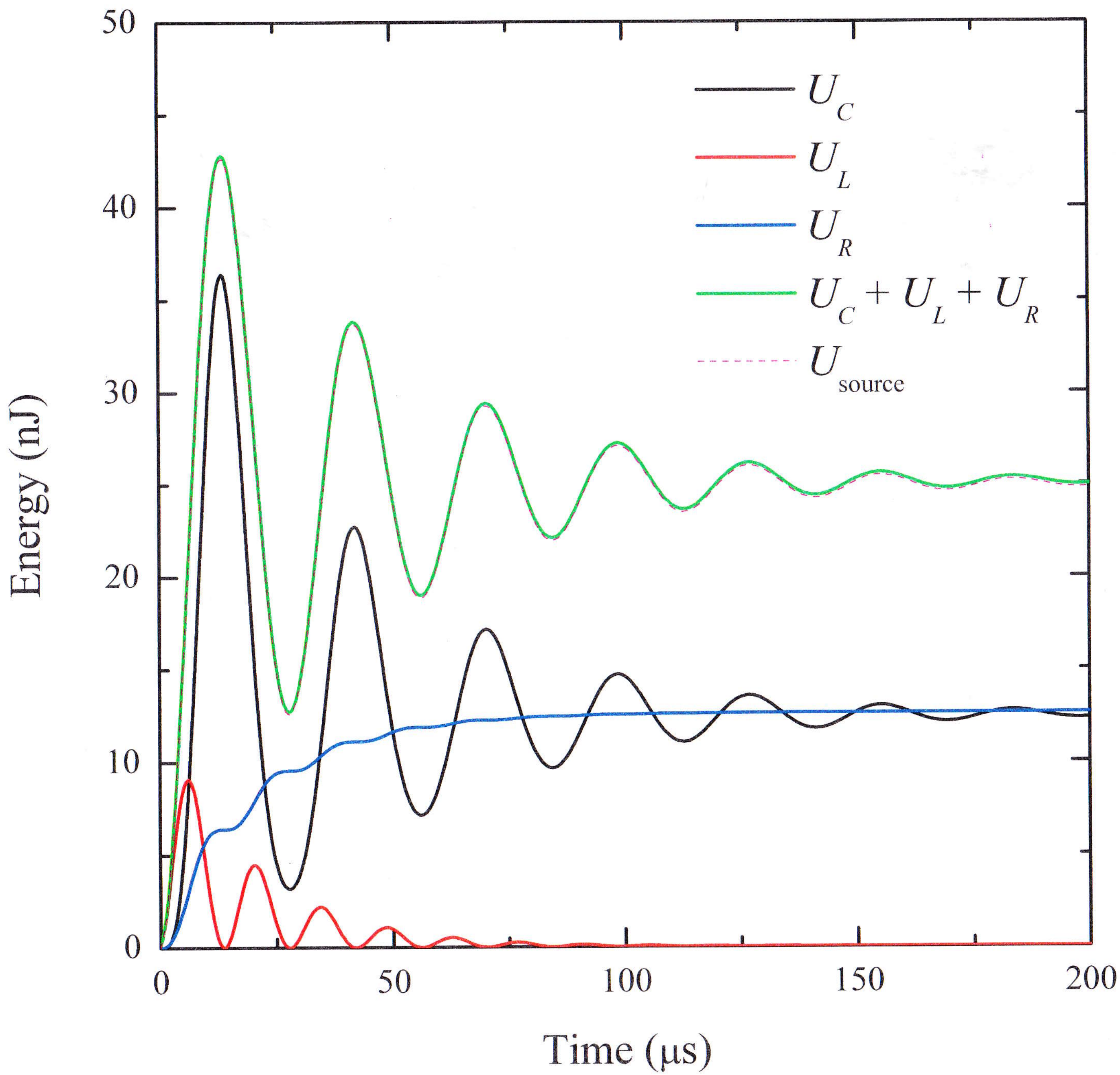
(16)

Series LRC-circuit for an initially uncharged capacitor with:
$R = 1\ \mathrm{k\Omega}$, $L = 20$ mH, $C = 1$ nF, and $V_0 = 5$ V (energy analysis)



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Return to viii & ix

$$m\frac{d^2x}{dt^2} + b\frac{dx}{dt} + kx = \underbrace{mg}_{F_G}$$

$$L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{1}{C}q = V_0$$

For mechanical system kinetic energy is $\frac{1}{2}mv^2 = \frac{1}{2}m\left(\frac{dx}{dt}\right)$

Analogous quantities in LRC circuit are

$$\left.\begin{array}{l} m \to L \\ v = \frac{dx}{dt} \to I = \frac{dq}{dt} \end{array}\right\} \therefore K.E = \frac{1}{2}mv^2$$

$$\Downarrow$$

$$\boxed{\frac{1}{2}LI^2 = U_L}$$

Magnetic energy stored by inductor corresponds to K.E. of mass.

When mass moves through equil. position, v is a max & K.E. peaks.

When charge on cap. goes through equil. charge, I is a max & U_L peaks.
(pos. or neg)

See plots of $V_R \propto I$ & U_L on page 18.

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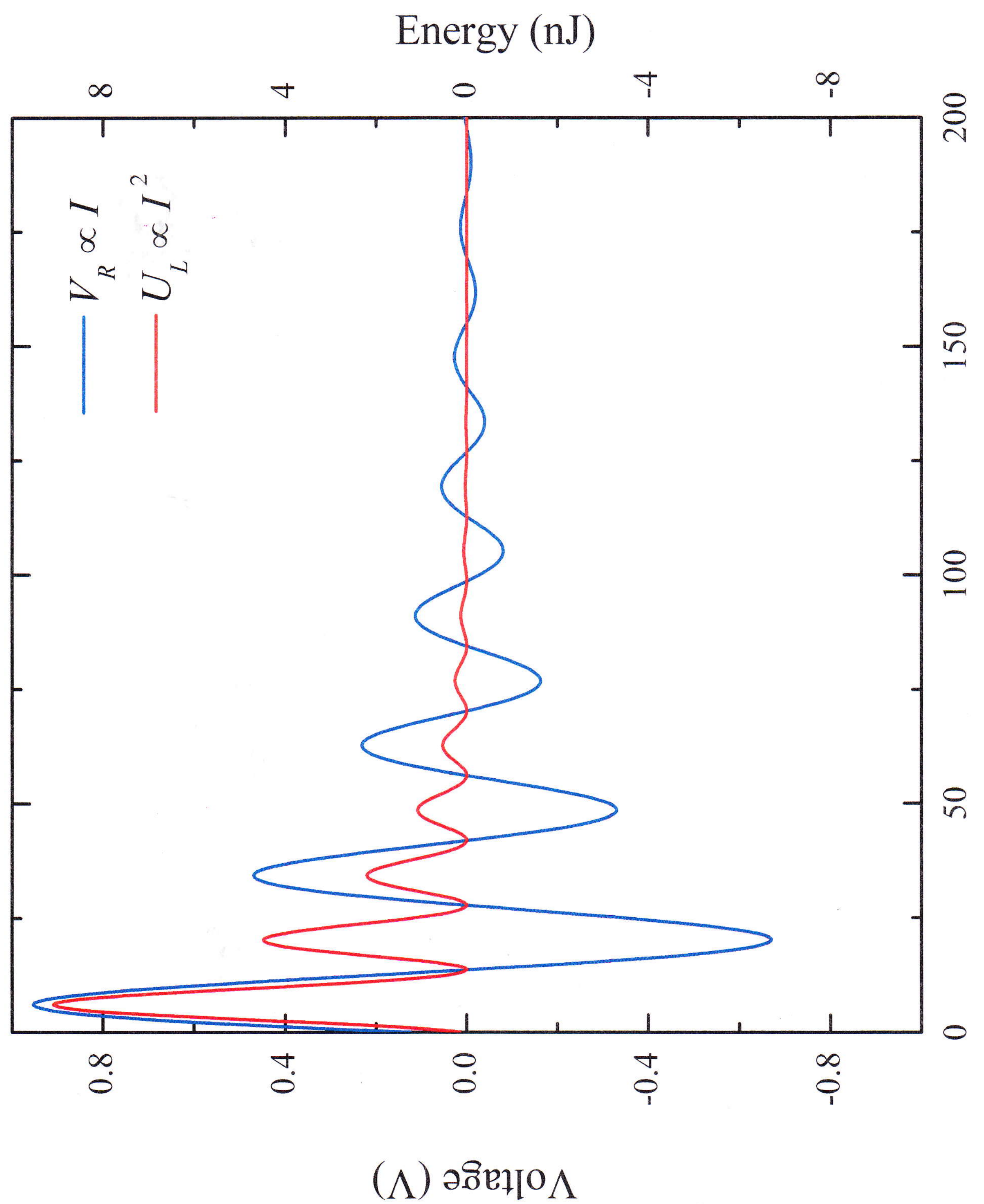
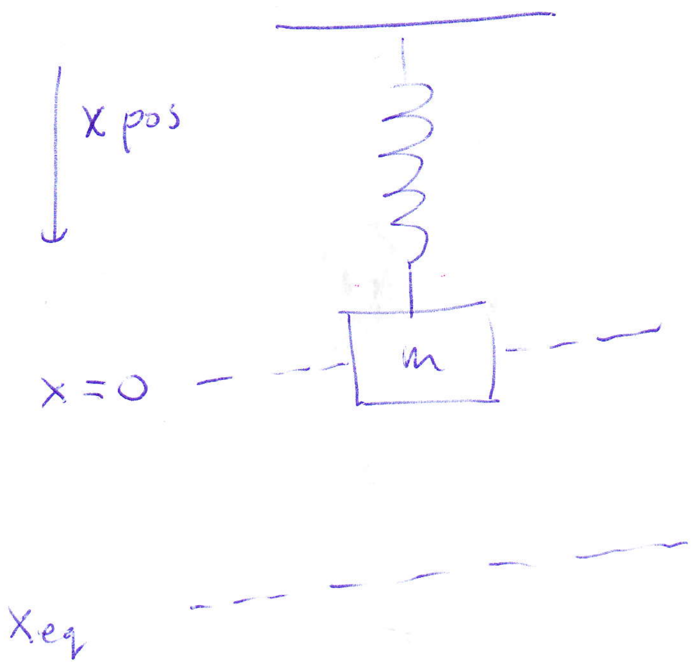


Series *LRC*-circuit for an initially uncharged capacitor with:
$R = 1\ \mathrm{k\Omega}$, $L = 20$ mH, $C = 1$ nF, and $V_0 = 5$ V

Consider P.E. of mech. system. Measure P.E. w.r.t $x=0$.



If could suddenly increase g, would establish new equil. pos. x_{eq}.

$$P.E. = \frac{1}{2}kx^2$$

Equil. P.E. is $\frac{1}{2}kx_{eq}^2$

$$= \frac{1}{2}k\left(\frac{mg}{k}\right)^2$$

$$= \frac{1}{2}\frac{F_a^2}{k}$$

Mechanical P.E. w/ oscillate about the value $\frac{F_a^2}{2k}$

Peaks & dips of P.E. correspond to where v is instantaneously zero (turning points).

In LRC circuit, $P.E. = \frac{1}{2}kx^2 \Rightarrow \frac{q^2}{2C} = U_C$.

When current is instantaneously zero ($V_R \propto I$), get peaks & dips in U_C.

See plots of $V_R \propto I$ & U_C on page 20.

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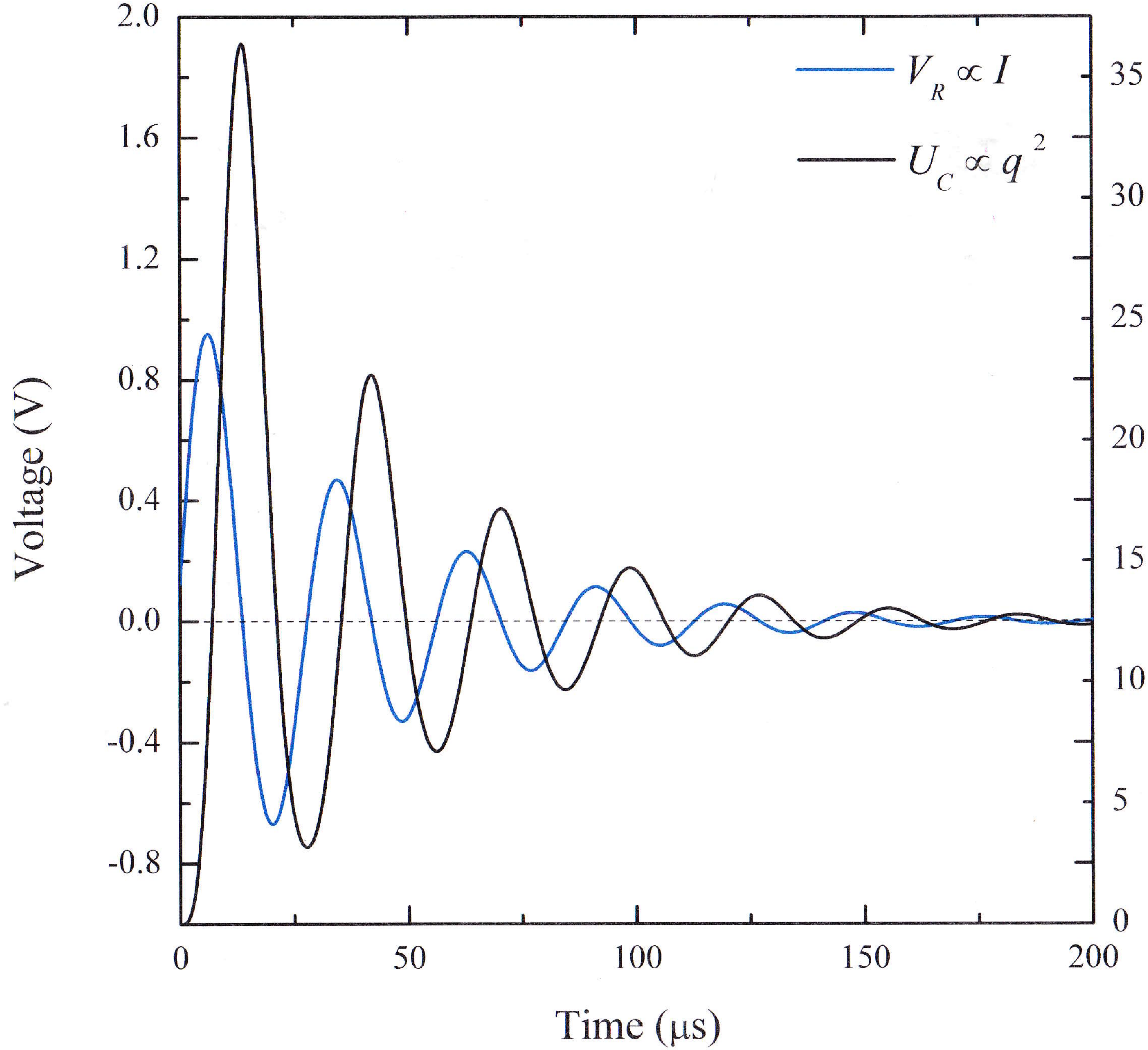
Series *LRC*-circuit for an initially uncharged capacitor with:
$R = 1\ \mathrm{k\Omega}$, $L = 20\ \mathrm{mH}$, $C = 1\ \mathrm{nF}$, and $V_0 = 5\ \mathrm{V}$



Finally power dissipated by resistor is $P_R = I^2 R$

$$\therefore U_R = \int I^2 R = R \int I^2 dt \Rightarrow b \int v^2 dt = \int (bv)\, v\, dt$$

$$= \int (bv)\left(\frac{dx}{dt} dt\right)$$

$$= \int D\, dx = \text{work by drag force.}$$

Power supplied by battery is $P_{source} = I \cdot V_0$

$$\therefore U_{source} = \int V_0 I\, dt = V_0 \int I\, dt \Rightarrow F_G \int v\, dt = \underbrace{F_G \Delta x}_{\text{work by force of gravity.}}$$

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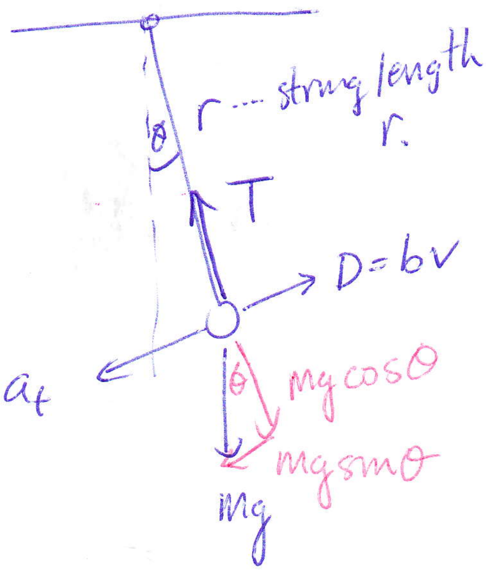
Now, just for fun, let's see if we can think of another mechanical analog $\Rightarrow$ pendulum.



$$m a_t = mg\sin\theta - bv$$

$$a_t = r\alpha = r\ddot{\theta} = r\frac{d^2\theta}{dt^2}$$

$$v = r\omega = r\dot{\theta} = r\frac{d\theta}{dt}$$

$$\therefore mr\frac{d^2\theta}{dt^2} + br\frac{d\theta}{dt} - mg\sin\theta = 0$$

small angle approx $\sin\theta \approx \theta$

$$m\frac{d^2\theta}{dt^2} + b\frac{d\theta}{dt} - \frac{mg}{r}\theta = 0$$

} like eq'ns (viii) & (ix) on page (11), but there is no constant (V_0 or F_G) on r.h.s.

Need a mechanism to shift equilibrium position.

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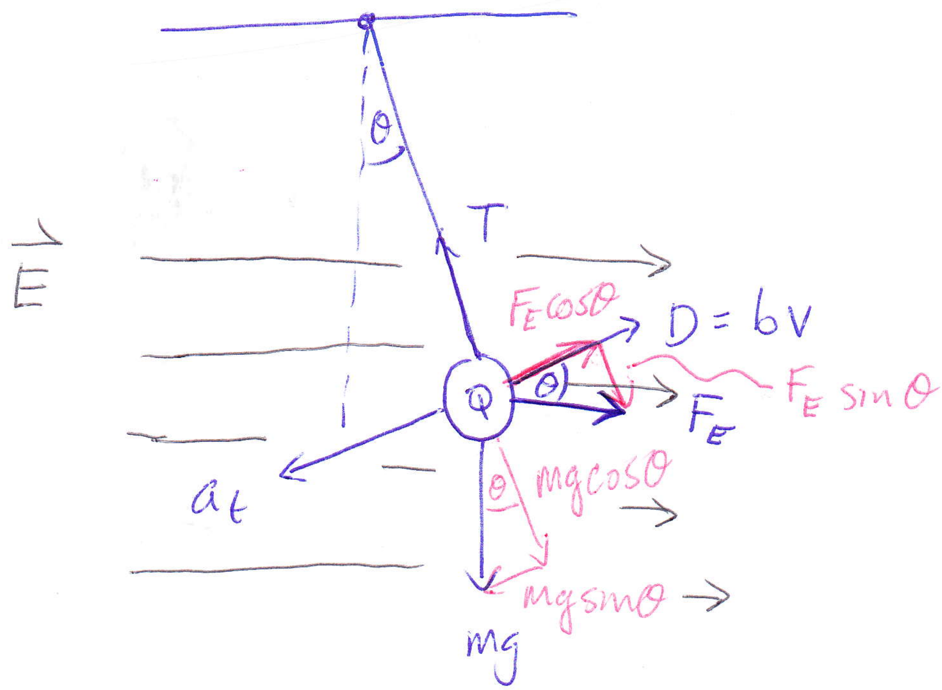
Consider the following modification, place charge Q on bob of the pendulum & apply a horizontal electric field.



Now
$$ma_t = mg\sin\theta - bv - F_E\cos\theta$$
$$mr\alpha = mg\sin\theta - br\omega - F_E\cos\theta$$
$$mr\frac{d^2\theta}{dt^2} = mg\sin\theta - br\frac{d\theta}{dt} - F_E\cos\theta$$

small angle approx $\quad \sin\theta \approx \theta$

$\cos\theta \approx 1$

$$\therefore\ mr\frac{d^2\theta}{dt^2} = mg\theta - br\frac{d\theta}{dt} - F_E$$

Finally,

$$\boxed{F_E = -mr\frac{d^2\theta}{dt^2} - br\frac{d\theta}{dt} + mg\theta}$$

Like viii & ix

Can shift equil. position of pendulum by tuning either the charge Q or the electric field $\vec{E}$

$(\vec{F}_E = Q\vec{E})$